

Comments on the floating body and the hyperplane conjecture

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To my dear sister Anna

ABSTRACT. We provide a reformulation of the hyperplane conjecture (the slicing problem) in terms of the floating body and give upper and lower bounds on the logarithmic Hausdorff distance between an arbitrary convex body $K \subset \mathbb{R}^d$ and the convex floating body K_δ inside K .

1. Introduction

Let $d \in \mathbb{N}$ and let $K \subset \mathbb{R}^d$ be a convex body (a compact convex set with non-empty interior). We shall index half-spaces as $\mathfrak{H}_{\theta,t} = \{x \in \mathbb{R}^d : \langle x, \theta \rangle \geq t\}$ where $\theta \in S^{d-1}$ and $t \in \mathbb{R}$. For any $\delta > 0$, the *convex floating body* inside K is defined as

$$K_\delta = \cap \{\mathfrak{H}_{\theta,t} : \text{vol}_d(\mathfrak{H}_{\theta,t} \cap K) \geq (1 - \delta)\text{vol}_d(K)\}$$

We shall refer to K_δ simply as the floating body, although this terminology is sometimes used for a non-convex variation of K_δ . The convex floating body was introduced by Schütt and Werner [19] and is a well studied object in convex geometry; it is related to Gauss-Kronecker curvature, affine surface area and polyhedral approximation ([19], [17], [18]) and plays an important role in the study of random polytopes (see e.g. [3] and [20] p. 1290). The definition of the convex floating body can be extended to an arbitrary probability measure on \mathbb{R}^d in the obvious way (see [9]) and is a very natural multivariate version of a quantile. In this paper we discuss a relationship between the floating body and the hyperplane conjecture.

The hyperplane conjecture (also known as the slicing problem) speculates that there exists a universal constant $c > 0$ such that for any $d \in \mathbb{N}$ and any convex body $K \subset \mathbb{R}^d$ of unit volume, there exists a hyperplane $\mathcal{H} \subset \mathbb{R}^d$ such that $\text{vol}_{d-1}(\mathcal{H} \cap K) \geq c$. This conjecture goes back to the 1986 paper of Bourgain [7] (see the remark on p.1470) and is one of the most fundamental unsolved problems in asymptotic convex geometry. It is equivalent to several other open problems in the area (see [16]). One such open problem is a variation of the Busemann-Petty problem [10]:

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Conjecture 1. *There exists a universal constant $c > 0$ with the following property: if $K, L \subset \mathbb{R}^d$ denote any centrally symmetric convex bodies in any dimension d such that for all central hyperplanes H we have*

$$\text{vol}_{d-1}(K \cap H) \leq \text{vol}_{d-1}(L \cap H)$$

then it follows that

$$\text{vol}_d(K) \leq c \cdot \text{vol}_d(L)$$

Let X be a random vector uniformly distributed inside a convex body K . We say that K is isotropic¹ if its centroid lies at the origin and the covariance of X obeys

$$\text{cov}(X) = L_K^2 I_d$$

where $L_K > 0$ is called the isotropic constant of K and I_d is the $d \times d$ identity matrix. Any convex body can be brought to isotropic position via an affine map (see e.g. [1] or [16]). It is well known that the hyperplane conjecture holds if and only if there exists a universal constant $c > 0$ such that $L_K < c$ for any isotropic convex body of unit volume in any dimension. The best bound to date is $L_K < cd^{1/4}$ by Klartag [13] (see also [8]). Let $B(0, r) = \{x \in \mathbb{R}^d : \|x\|_2 \leq r\}$ and consider the following two statements:

- Λ_1 : there exists $\delta \in (0, e^{-1})$ and $r > 0$ such that for all $d \in \mathbb{N}$ and all isotropic convex bodies $K \subset \mathbb{R}^d$ of unit volume, $K_\delta \subset B(0, r)$.
- Λ_2 : for all $\delta \in (0, e^{-1})$ there exists $r > 0$ such that for all $d \in \mathbb{N}$ and all isotropic convex bodies $K \subset \mathbb{R}^d$ of unit volume, $K_\delta \subset B(0, r)$.

Theorem 1. *The hyperplane conjecture is equivalent to both Λ_1 and Λ_2 .*

The proof of theorem 1 essentially comes down to an observation that due to the rigidity of one dimensional log-concave probability distributions, the variance of such a distribution is controlled by certain quantiles (and vice versa). Thus the condition of bounded variance ($L_K < c$) equates to a bound on the quantiles, and this easily transfers to the statements Λ_1 and Λ_2 concerning the floating body.

We define the *logarithmic Hausdorff distance* between two convex bodies $K, L \subset \mathbb{R}^d$ as

$$d_{\mathfrak{L}}(K, L) = \inf\{\lambda \geq 1 : \exists x \in \text{int}(K \cap L), \lambda^{-1}(K - x) + x \subset L \subset \lambda(K - x) + x\}$$

The logarithmic Hausdorff distance compares both the size and the shape of the two bodies and is related to both the Hausdorff distance $d_{\mathcal{H}}$ and the Banach-Mazur distance d_{BM} . Unlike the Hausdorff distance, it is invariant under affine transformations that act simultaneously on both bodies, yet it is stronger than the Banach-Mazur distance which is blind to affine transformations that act on one body but not the other. We end the paper with the following two theorems. The bound (1.1) is used in [9].

Theorem 2. *For all $d \in \mathbb{N}$, all convex bodies $K \subset \mathbb{R}^d$ and all $\delta \leq 8^{-d}$, the body K_δ is non-empty and we have the inequalities*

$$(1.1) \quad d_{\mathfrak{L}}(K, K_\delta) \leq 1 + 8\delta^{1/d}$$

$$(1.2) \quad d_{BM}(K, K_\delta) \leq 1 + 24\delta^{1/d}$$

¹Note that the word isotropic is used to mean several different things in the literature. Some authors require $L_K = 1$, others require $\text{vol}_d(K) = 1$. Our use of the word does not include either of these conditions.

By (4.4), inequality (1.1) is sharp (except for the constant 8) when K is a simplex.

Theorem 3. *There exists a universal constant $c > 0$ such that for all $d \in \mathbb{N}$, all convex bodies $K \subset \mathbb{R}^d$ and all $\delta \in (0, e^{-1})$, the body K_δ is non-empty and*

$$(1.3) \quad d_{\mathfrak{L}}(K, K_\delta) \geq \frac{cd^{\frac{1}{4}}}{\log(2\delta^{-1})}$$

2. Preliminaries

For any convex body $K \subset \mathbb{R}^d$ containing the origin as an interior point, we consider the dual Minkowski functional $\|\cdot\|_{K^\circ}$ defined by $\|y\|_{K^\circ} = \sup\{\langle y, x \rangle : x \in K\}$. Let $h_K : S^{d-1} \rightarrow \mathbb{R}$ denote the support function of K , which is the restriction of $\|\cdot\|_{K^\circ}$ to S^{d-1} . For any $\theta \in S^{d-1}$ and $t \in \mathbb{R}$, let $\mathcal{H}_{\theta,t} = \{y \in \mathbb{R}^d : \langle \theta, y \rangle = t\}$. Let $\psi_{K,\theta}$ denote the normalized parallel section function of K in the direction of θ ,

$$\psi_{K,\theta}(t) = \frac{\text{vol}_{d-1}(K \cap \mathcal{H}_{\theta,t})}{\text{vol}_d(K)}$$

and define

$$A_{K,\theta}(t) = \int_t^{h_K(\theta)} \psi_{K,\theta}(s) ds$$

where in both formulae $t \in [-h_K(-\theta), h_K(\theta)]$. If X is a random vector uniformly distributed in K , then $\psi_{K,\theta}$ is the density function of $\langle X, \theta \rangle$. We denote the median of this random variable by

$$m_{K,\theta} = A_{K,\theta}^{-1}(1/2)$$

The *Banach-Mazur distance* between convex bodies K and L shall be denoted by $d_{BM}(K, L)$ and is defined as,

$$d_{BM}(K, L) = \inf\{\lambda \geq 1 : \exists x \in \mathbb{R}^d, \exists T, K \subset TL \subset \lambda(K - x) + x\}$$

where T represents an affine transformation of \mathbb{R}^d . By convexity we can express the Hausdorff distance between K and L as

$$d_{\mathcal{H}}(K, L) = \sup_{\theta \in S^{d-1}} \left| \sup_{x \in K} \langle \theta, x \rangle - \sup_{x \in L} \langle \theta, x \rangle \right|$$

We define the *logarithmic Hausdorff distance* between K and L about a point $x \in \text{int}(K \cap L)$ as

$$d_{\mathfrak{L}}(K, L, x) = \inf\{\lambda \geq 1 : \lambda^{-1}(K - x) + x \subset L \subset \lambda(K - x) + x\}$$

and

$$d_{\mathfrak{L}}(K, L) = \inf\{d_{\mathfrak{L}}(K, L, x) : x \in \text{int}(K \cap L)\}$$

Note that

$$\log d_{\mathfrak{L}}(K, L, 0) = \sup_{\theta \in S^{d-1}} |\log \|\theta\|_K - \log \|\theta\|_L|$$

The following relations follow directly from the definitions above,

$$\begin{aligned} d_{\mathfrak{L}}(K, L, 0) &= d_{\mathfrak{L}}(K^\circ, L^\circ, 0) \\ d_{BM}(K, L) &\leq d_{\mathfrak{L}}(K, L)^2 \end{aligned}$$

3. The hyperplane conjecture

A function $f : \mathbb{R}^d \rightarrow [0, \infty)$ is log-concave (see [14] and [15]) if for any $x, y \in \mathbb{R}^d$ and any $\lambda \in (0, 1)$ we have

$$f(\lambda x + (1 - \lambda)y) \geq f(x)^\lambda f(y)^{1-\lambda}$$

The support of such a function will necessarily be convex, and $-\log f$ is a convex function. A probability measure is log-concave if for any compact sets $\Omega_1, \Omega_2 \subset \mathbb{R}^d$ and any $\lambda \in (0, 1)$ the following inequality holds,

$$\mu(\lambda\Omega_1 + (1 - \lambda)\Omega_2) \geq \mu(\Omega_1)^\lambda \mu(\Omega_2)^{1-\lambda}$$

where $\Omega_1 + \Omega_2 = \{x + y : x \in \Omega_1, y \in \Omega_2\}$ is the Minkowski sum of Ω_1 and Ω_2 . The Brunn-Minkowski inequality (in its multiplicative form, see [2]) is the statement that Lebesgue measure is log-concave, while this was generalized in 1975 by Borell [6] who proved that an absolutely continuous probability measure is log-concave if and only if its density is log-concave. Perhaps the most important log-concave functions are the indicator functions of convex bodies. Log-concave functions are not only a functional generalization of convex bodies, they are of paramount importance in the study of convex bodies. An excellent example of this is Klartag's proof of the central limit theorem for convex bodies [12]. One of the key properties of log-concave measures is that the measure projection of a log-concave measure onto a linear subspace is log-concave (this is a consequence of the Prékopa-Leindler inequality, see [2]). In particular, if x is a random vector in \mathbb{R}^d with a log-concave distribution and $y \in \mathbb{R}^d$ is any fixed vector, then $\langle x, y \rangle$ has a log-concave distribution in \mathbb{R} . Hence, results on log-concave measures often reduce to the one dimensional case.

Just as for a convex body, a log-concave probability measure μ is called isotropic if its centroid lies at the origin and its covariance matrix is of the form

$$\text{cov}(\mu) = L_\mu^2 I_d$$

Log-concave measures are very rigid. A classic example of this rigidity (see lemma 5.5. in [15] and p. 1913 in [4]) is the following:

Lemma 1. *For any log-concave probability density function f defined on \mathbb{R} with mean zero and variance one,*

$$(3.1) \quad \begin{aligned} 1/8 &\leq f(0) \leq \sup_{t \in \mathbb{R}} f(t) \leq 1 \\ \frac{1}{\sqrt{12}} &\leq f(m) \leq \frac{1}{\sqrt{2}} \end{aligned}$$

where m is the median of f .

A direct consequence of the above lemma is that if μ is an isotropic log-concave probability measure on \mathbb{R}^d with density f and \mathcal{H} is any hyperplane containing the origin, then

$$(3.2) \quad \frac{1}{8} L_\mu^{-1} \leq \int_{\mathcal{H}} f(x) dx \leq L_\mu^{-1}$$

In particular, all central sections have roughly the same volume and the specific hyperplane \mathcal{H} mentioned in the hyperplane conjecture is actually irrelevant to the problem. It also demonstrates the equivalence between the hyperplane conjecture and the claim $L_K < c$. To see why the bound (3.2) follows from (3.1), consider the

case $L_\mu = 1$ and let X be a random vector with distribution μ . For any $\theta \in S^{d-1}$ the random variable $\langle X, \theta \rangle$ has mean zero, variance one and a density given by

$$f_\theta(t) = \int_{\langle x, \theta \rangle = t} f(x) dx$$

Using similar reasoning and a higher dimensional version of (3.1), see e.g. lemma 5.14 in [15], it is easy to prove that if \mathcal{H} is any affine subspace of dimension k containing the centroid of μ then

$$c_1(n, k) L_\mu^{-(n-k)} \leq \int_{\mathcal{H}} f(x) dx \leq c_2(n, k) L_\mu^{-(n-k)}$$

where $c_1(n, k) = 2^{-7(n-k)}$ and $c_2(n, k) = (n-k)(20(n-k))^{(n-k)/2}$. This is a slight generalization of a result by Hensley [11] (see also the discussion in [1] and [5]).

Another rigidity property (see lemma 5.4 in [15]) is that for any log-concave probability density function f defined on \mathbb{R} with mean zero,

$$e^{-1} \leq \int_0^\infty f(x) dx \leq 1 - e^{-1}$$

Equivalently $F^{-1}(1-e) \geq 0$, where F is the cumulative distribution corresponding to f . Piecing together known results, we easily prove the following extension:

Lemma 2. *Let $\rho \in (0, e^{-1})$ and let μ be an absolutely continuous log-concave probability measure on \mathbb{R} with mean zero, variance σ^2 and cumulative distribution F . Then,*

$$(3.3) \quad (e^{-1} - \rho)\sigma \leq F^{-1}(1 - \rho) \leq 10 \log(2\rho^{-1})\sigma$$

PROOF. We can assume without loss of generality that $\sigma = 1$. By the result cited above (lemma 5.4 in [15]), $F^{-1}(1 - \rho) > 0$. Let x be a random variable with distribution μ and density f . Lemma 5.5 in [15] states that $f(0) \geq 1/8$ and for all $t \in \mathbb{R}$, $f(t) \leq 1$. Hence (again by lemma 5.4 in [15])

$$\begin{aligned} e^{-1} - \rho &\leq \mu\{y \in \mathbb{R} : 0 \leq y \leq F^{-1}(1 - \rho)\} \\ &= \int_0^{F^{-1}(1-\rho)} f(s) ds \\ &\leq F^{-1}(1 - \rho) \end{aligned}$$

Since $g = -\log f$ is convex, $f = e^{-g}$ decays exponentially (or quicker). In particular (see e.g. lemma 2.2. in [12]) for all $t \geq 0$,

$$1 - F(t) \leq 2e^{-t/10}$$

from which the upper bound follows. \square

This easily transfers to the multidimensional setting. For any half-space \mathfrak{H} containing the centroid of a log-concave probability measure μ , $\mu(\mathfrak{H}) \geq e^{-1}$ (see lemma 5.12 in [15]). Equivalently, any half-space of mass less than e^{-1} can not contain the centroid. The above lemma can be re-cast as follows.

Lemma 3. *Let $\rho \in (0, e^{-1})$ and let μ be an isotropic log-concave probability measure on \mathbb{R}^d . If $\mathfrak{H}_{\theta,t} = \{x \in \mathbb{R}^d : \langle \theta, x \rangle \geq t\}$ is any half-space with $\mu(\mathfrak{H}_{\theta,t}) = \rho$ then*

$$(3.4) \quad (e^{-1} - \rho)L_\mu \leq t \leq 10 \log(2\rho^{-1})L_\mu$$

PROOF OF THEOREM 1. Denote the hyperplane conjecture as $\Lambda_{\mathcal{H}}$. Clearly $\Lambda_2 \Rightarrow \Lambda_1$. We now demonstrate that $\Lambda_1 \Rightarrow \Lambda_{\mathcal{H}} \Rightarrow \Lambda_2$.

Suppose that Λ_1 is true and let $\delta' = (\delta + e^{-1})/2$. Consider any $d \in \mathbb{N}$ and any isotropic convex body K of unit volume. By (3.4), if $\mathfrak{H}_{\theta,t} = \{x \in \mathbb{R}^d : \langle \theta, x \rangle \geq t\}$ is any half-space with $\text{vol}_d(\mathfrak{H}_{\theta,t} \cap K) = \delta$, then $t \geq L_{\mu}(e^{-1} - \delta)$. Hence $\mathfrak{H}_{\theta,t} \cap B(0, L_{\mu}(e^{-1} - \delta')) = \emptyset$. Since this holds for any such hyperplane, $B(0, L_{\mu}(e^{-1} - \delta')) \subset K_{\delta}$. By Λ_1 , $K_{\delta} \subset B(0, r)$. Hence $L_{\mu} \leq r(e^{-1} - \delta')^{-1}$. Since this holds for any isotropic convex body in any dimension, this implies the truth of $\Lambda_{\mathcal{H}}$.

Suppose that $\Lambda_{\mathcal{H}}$ is true. In particular, $L_K < c$ for any convex body K . Consider any $\delta \in (0, e^{-1})$ and let $r = 10 \log(2\delta^{-1})c$. Let K be an isotropic convex body of unit volume in \mathbb{R}^d and let $\mathfrak{H}_{\theta,t} = \{x \in \mathbb{R}^d : \langle \theta, x \rangle \geq t\}$ denote any half-space with $\text{vol}_d(\mathfrak{H}_{\theta,t} \cap K) = \delta$. By (3.4), $t \leq r$. For any $\theta \in S^{n-1}$ such a half-space exists, hence $K_{\delta} \subset B(0, r)$ and Λ_2 holds true. \square

4. Small perturbations

PROOF OF THEOREM 2. We may assume without loss of generality that K is isotropic. In particular, the center of mass of K is zero. Fix any $\theta \in S^{d-1}$ and $t \in [m_{\theta}, h_K(\theta)]$. Brunn's theorem [2] claims that the function $s \mapsto \psi_{\theta}(s)^{1/(d-1)}$ is concave on its support. For any $s \in (t, h_K(\theta))$, we have the convex combination

$$s = \frac{h_K(\theta) - s}{h_K(\theta) - t}t + \frac{s - t}{h_K(\theta) - t}h_K(\theta)$$

Since $\psi_{\theta} \geq 0$, it follows by concavity that

$$\psi_{\theta}(s) \geq \left(\frac{h_K(\theta) - s}{h_K(\theta) - t} \right)^{d-1} \psi_{\theta}(t)$$

and by integration that

$$(4.1) \quad A_{\theta}(t) \geq \alpha_{\theta}(t)\psi_{\theta}(t)$$

where $\alpha_{\theta}(t) = d^{-1}(h_K(\theta) - t) > 0$. However, for any $s \in (m_{\theta}, t)$ we have the convex combination

$$t = \frac{h_K(\theta) - t}{h_K(\theta) - s}s + \frac{t - s}{h_K(\theta) - s}h_K(\theta)$$

We again have by concavity that

$$\psi_{\theta}(t) \geq \left(\frac{h_K(\theta) - t}{h_K(\theta) - s} \right)^{d-1} \psi_{\theta}(s)$$

hence

$$\begin{aligned} 1/2 - A_{\theta}(t) &= \int_{m_{\theta}}^t \psi_{\theta}(s) ds \\ &\leq \beta_{\theta}(t)\psi_{\theta}(t) \end{aligned}$$

where $\beta_{\theta}(t) = d^{-1}[h_K(\theta) - t]^{1-d}([h_K(\theta) - m_{\theta}]^d - [h_K(\theta) - t]^d) > 0$. Thus,

$$(4.2) \quad A_{\theta}(t) \geq 1/2 - \beta_{\theta}(t)\psi_{\theta}(t)$$

For large values of $\psi_\theta(t)$, (4.1) is a better bound, while for small values of $\psi_\theta(t)$ (4.2) is better. Minimizing the function $u \mapsto \max\{\alpha u; 1/2 - \beta u\}$ gives

$$\begin{aligned} A_\theta(t) &\geq \frac{1}{2} \alpha_\theta(t) [\alpha_\theta(t) + \beta_\theta(t)]^{-1} \\ (4.3) \quad &= \frac{1}{2} \left(\frac{h_K(\theta) - t}{h_K(\theta) - m_\theta} \right)^d \end{aligned}$$

Let $t_\theta = A_\theta^{-1}(\delta)$. Note that $\text{vol}_d(K \cap \mathfrak{H}_{\theta, t_\theta}) = \delta$ and that $h_K(\theta) - m_\theta \leq 2h_K(\theta)$. By (4.3),

$$\begin{aligned} t_\theta &\geq h_K(\theta) - (2\delta)^{1/d} (h_K(\theta) - m_\theta) \\ &\geq h_K(\theta) (1 - 4\delta^{1/d}) \end{aligned}$$

This implies that $\mathfrak{H}_{\theta, t_\theta} \cap (1 - 4\delta^{1/d})K = \emptyset$. Since this holds for all $\theta \in S^{d-1}$, we have the inclusion $(1 - 4\delta^{1/d})K \subset K_\delta$, and the bounds (1.1) and (1.2) hold. \square

Inequality (1.1) is essentially sharp. To see this, consider the simplex in standard orthogonal position,

$$\Delta_c^d = \{x \in \mathbb{R}^d : \forall i \ x_i \geq 0, \sum_{i=1}^d x_i \leq 1\}$$

and the half-space $\mathfrak{H} = \{x \in \mathbb{R}^d : x_1 \leq 1 - \delta^{1/d}\}$. Since $\Delta_c^d \setminus \mathfrak{H}$ is homothetic to Δ_c^d by a factor of $\delta^{1/d}$, $\text{vol}_d(\mathfrak{H} \cap \Delta_c^d) = (1 - \delta)\text{vol}_d(\Delta_c^d)$. Hence, if $K = \Delta_c^d$, then $K_\delta \subset \mathfrak{H}$ and

$$(4.4) \quad d_{\mathfrak{L}}(K, K_\delta) \geq 1 + \frac{1}{2}\delta^{1/d}$$

5. Large perturbations

In high dimensional spaces, the mass inside an isotropic convex body K is roughly normally distributed in most directions. The measure of 'most directions' is Haar measure on S^{d-1} . For the Euclidean ball this is referred to as Maxwell's principal, while the general case was proved by Klartag [12]. A consequence of this is that most of the one dimensional marginals of the uniform probability measure on K have tails that are very long and very light. Even to cut off a small proportion of the mass of K in a fixed direction requires us to go very deeply inside the body. Thus the body K_δ may be much smaller than the original body K .

Indeed, as was pointed out to myself by Professor Mathieu Meyer, for fixed $\delta > 0$ the effect of the operation $K \mapsto K_\delta$ is, asymptotically in dimension, worst possible. By a result in [16], for all $\delta > 0$ there exists a universal constant $C > 1$ such that for any $d \in \mathbb{N}$ and any centrally symmetric convex body $K \subset \mathbb{R}^d$, $d_{BM}(K_\delta, B_2^d) \leq C$. Hence, there will always be some symmetric convex body K (in particular the cube B_∞^d and the cross polytope B_1^d) such that $d_{BM}(K, K_\delta) \approx \sqrt{d}$. Both the size and the shape of K are dramatically altered.

PROOF OF THEOREM 3. The relationship between K and K_δ is independent of Euclidean structure and we can therefore assume that K is in isotropic position. Define $v_d = \text{vol}_d(B_2^d) = \pi^{d/2}(\Gamma(\frac{d}{2} + 1))^{-1}$. Hence the body $v_d^{-1/d}B_2^d$ has unit volume. By Stirling's formula, $v_d^{-1/d} \geq c'\sqrt{d}$. Since $\text{vol}_d(K) = 1$, it can not be

a proper subset of $v_d^{-1/d}B_2^d$, and there exists $y \in K$ with $\|y\|_2 \geq c'\sqrt{d}$. As in the proof of theorem 1, we have $K_\delta \subset 10L_K \log(2\delta^{-1})B_2^d \subset cd^{1/4} \log(2\delta^{-1})B_2^d$. Inequality (1.3) then follows. \square

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